Ergodic Theory and Measured Group Theory
Lecture 7

Full proof of the ptwise erg. thm.

— $\int f \, d\mu = 0$

— $\limsup_n A_n f$ constant a.e.

- Suppose towards a contradiction that $f^* := \min\{\frac{1}{2}\limsup_n A_n f, 1\} > 0$.
- Denote $c := \int_X f^* \, d\mu = f^*$.
- Define $x \mapsto n(x) : X \to \mathbb{N}$, where $n(x) =$ the least $n \in \mathbb{N}$ s.t. $A_n f(x) \geq f^*$.
- $\exists$ large enough $M > 0$ s.t. $X \setminus Y := f^{-1}(-\infty, -M) \subseteq X$ has small enough measure so that $\int_{X \setminus Y} |f - f^*| \, d\mu < \frac{c}{3}$, in particular, $\int_{X \setminus Y} f - f^* > -\frac{c}{3}$.

Now we'll deal with $\mathbb{1}_Y f$ instead of f.

- Apply the tiling lemma to $x \mapsto n(x)$ with $\varepsilon = \frac{c}{2 \cdot (M+1) \cdot 3}$ and N, and get that $\exists$ set Z of measure $\geq 1 - \varepsilon$ s.t. $\forall x \in Z$, the interval $I_N \cdot x$ is tiled, up to an $\leq \varepsilon$-fraction, with tiles of the form $I_{n(y)} \cdot y$. Note that $A_{n(y)} f \geq f^*$, hence $A_{n(y)}(f - f^*) \geq 0$, but also $\mathbb{1}_Y (f - f_*) \geq f - f_*$, so $A_{n(y)}(\mathbb{1}_Y (f - f^*)) \geq 0$. Therefore, $\forall x \in Z$,

$$A_N(\mathbb{1}_Y (f - f^*))(x) \geq (1-\varepsilon) \cdot 0 + \varepsilon \cdot (-M-1) = -(M+1) \cdot \varepsilon.$$

— $\displaystyle\int_Y (f-f^*)\,d\mu = \int_X \mathbb{1}_Y (f-f^*)\,d\mu \underset{\text{bridge}}{=} \int_X A_N(\mathbb{1}_Y(f-f^*))\,d\mu = \int_Z A_N(\mathbb{1}_Y(f-f^*))\,d\mu$

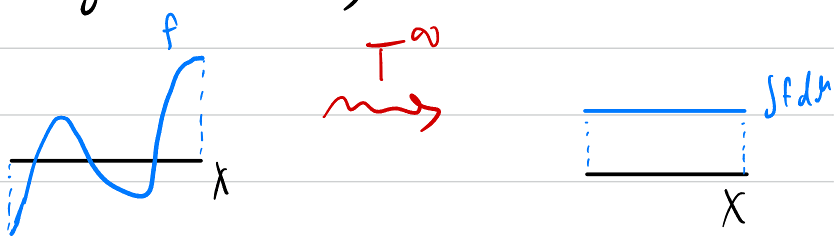
$$+\int_{X\setminus Z} A_N(\mathbb{1}_Y(f-f^*))\,d\mu \geq -(M+1)\varepsilon\cdot\mu(Z) - (M+1)\cdot\mu(X\setminus Z)$$

$$\geq -(M+1)\varepsilon - (M+1)\cdot\varepsilon = -2(M+1)\cdot\varepsilon.$$

— Hence, $\displaystyle -c = \int f\,d\mu - c = \int_X (f-f^*)\,d\mu = \int_Y (f-f^*)\,d\mu + \int_{X\setminus Y}(f-f^*)\,d\mu$

$$\geq -2\cdot(M+1)\cdot\varepsilon - \frac{c}{3} \geq -\frac{2c}{3},$$ a contradiction. $\square$

Ptwise ergodic for nonergodic. Intuitively, for an ergodic pmp T, the pointwise ergodic theorem says is that T stirs up the whole X so well that it spreads any L^1-function f evenly on X making it the constant $\int f\,d\mu$.



For a nonergodic T, we know that $\exists$ partition $X = X_1 \sqcup X_2$, s.t. T never moves any point of X_1 into X_2, and vice versa, w/ $\mu(X_1), \mu(X_2) > 0$. The best one can hope is that

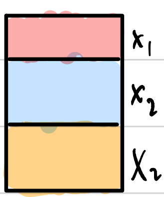
T^∞ spreads $\mathbb{1}_{X_i} \cdot f$ over X_i evenly. If X is partitioned "enough" into invariant pieces, then is true and the piecewise evened out version of f is called the **conditional expectation** of f with respect to the σ-algebra of Borel T-invariant sets.

Def. Let $\mathcal{A}$ be a sub-σ-algebra of the Borel σ-algebra $\mathcal{B}(X)$ and let $f \in L^1(X, \mu)$. An integrable function $\bar{f}$ measurable w.r.t. $\mathcal{A}$ is called the **conditional expectation** of f w.r.t. $\mathcal{A}$ if $\forall A \in \mathcal{A}$,

$$\int_A f \, d\mu = \int_A \bar{f} \, d\mu .$$

Offical notation is $\mathbb{E}(f \mid \mathcal{A})$.

Example. Let $X = X_1 \sqcup X_2 \sqcup X_3$ & let $\mathcal{A}$ be the σ-alg generated by $\{X_1, X_2, X_3\}$. The $\mathcal{A}$-measurable functions are exactly those constant on each X_i.



Then $\bar{f}|_{X_i} \equiv \int_{X_i} f \, d\mu$, for $i = 1, 2, 3$.

Def. For $\mathcal{A}$ as in the def above, define eq. rel. $E_{\mathcal{A}}$ on X by,

$$x \, E_{\mathcal{A}} \, y \; :\Longleftrightarrow \; \forall A \in \mathcal{A}, \ (x \in A \Leftrightarrow y \in A).$$

Using the analytic separation theorem from descriptive set theory, one can show that if $\mathcal{A} \subseteq \mathcal{B}(X)$ is ctbly-generated, then the $\mathcal{A}$-measurable functions are precisely those Borel functions that are $E_{\mathcal{A}}$-invariant (i.e. constant on every $E_{\mathcal{A}}$-class).

Theorem. $\forall f \in L^1(X,\mu)$, $\mathbb{E}(f \mid \mathcal{A})$ exists.

Proof. Suppose $f \geq 0$. Let μ_f be the measure given by $d\mu_f = f d\mu$, i.e. $\forall B \subseteq X$, $\mu_f(B) = \int_B f \, d\mu$. Let μ_f' and μ' be the restrictions of μ_f and μ to the σ-algebra $\mathcal{A}$. But still, $\mu_f' \ll \mu'$, so $\exists$ Radon-Nikodym derivative $\bar{f} := \frac{d\mu_f'}{d\mu'}$, which by def. is an $\mathcal{A}$-measurable function. Check that $\bar{f}$ is $\mathbb{E}(f \mid \mathcal{A})$. $\square$

Remark. For $f \in L^2(X,\mu)$, $\mathbb{E}(f \mid \mathcal{A})$ is the closest (i.e. the projection in the Hilbert space $L^2(X,\mu)$) to f among all $\mathcal{A}$-measurable functions.

Ptwise ergodic for general T. Let T be a pmp transformation on (X,μ). $\forall f \in L^1(X,\mu)$,

$$\lim_n A_n^T f = \mathbb{E}(f \mid \mathcal{B}_T) \quad \text{a.e.}$$

where $\mathcal{B}_T$ is the σ-algebra of all T-invariant Borel sets.

Proof. is the same as for ergodic T, just keep in mind that f^* is only a T-inv function (not necessarily constant). $\square$

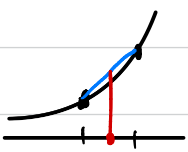
L^p-ergodic theorem. Let (X,μ) be a st. prob. space.

Properties of $\mathbb{E}(f|\mathcal{A})$. For any sub-$\sigma$-alg. $\mathcal{A} \subseteq \mathcal{B}(X)$, the conditional expectation is an operator from $L^p(X,\mu)$ to $L^p(X,\mu)$, which is a contraction, i.e.

$$\|\mathbb{E}(f|\mathcal{A})\|_p \leq \|f\|_p, \quad \forall f \in L^p(X,\mu),$$

for all $p \geq 1$.

Proof. This follows from Jensen's inequality (i.e. convexity of $(\,)^p$),



i.e. the p^{th} power of an average $\leq$ the average of p^{th} powers. $\boxtimes$

This is similar to the averaging operator A_n^T...

Bridge lemma*. Let $n \in \mathbb{N}$, T a pmp trans. on (X,μ), $f \in L^1(X,\mu)$.

(a) $\int_B f \, d\mu = \int_B A_n f \, d\mu$ for every T-invariant set B.

(b) $\|A_n f\|_p \leq \|f\|_p$ for all $p \geq 1$, given $f \in L^p(X,\mu)$.

Proof. (a) By the usual bridge lemma for $\mathbb{1}_B$,

$$\int_B f\,d\mu = \int_X \mathbb{1}_B f\,d\mu = \int_X A_n \mathbb{1}_B f\,d\mu = \int_B A_n f\,d\mu.$$

(b) $$\|A_n f\|_p^p = \int |A_n f|^p\,d\mu \underset{\Delta\text{-ineq.}}{\leq} \int (A_n|f|)^p\,d\mu \underset{\text{convexity of } ()^p}{\leq}$$

$$\leq \int A_n(|f|^p)\,d\mu \underset{\text{bridge}}{=} \int |f|^p\,d\mu = \|f\|_p^p.$$

$\square$